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Representations of $\mathcal{U}_q(sl(N))$ at roots of unity

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Abstract. The Gelfand–Zetlin basis for representations of $\mathcal{U}_q(sl(N))$ is improved to better fit the case when q is a root of unity. The usual q-deformed representations, as well as the nilpotent, periodic (cyclic), semi-periodic (semi-cyclic) and also some atypical representations are now described with the same formalism.

1. Introduction

We are interested in quantum Lie algebras [1–3] and their finite dimensional irreducible representations. At generic deformation parameter q, the classification of irreducible representations is in correspondence with the classical case [4]. When q is a mth root of unity, there are two options:

One can consider first the restricted quantum Lie algebra, where the raising and lowering generators are nilpotent, i.e. $e_\alpha^m = f_\alpha^m = 0$ and where the Cartan generators h_i are such that $k_i^m = (q^{h_i})^m = 1$. Representations of these were studied by Lusztig [5]. A classification of irreducible representations of $\mathcal{U}_q(sl(3))$ was done in [6].

The other option is to fix no relation for the mth powers of these generators, which are actually, for an odd value of m, in the centre of the quantum algebra [7]. Then the irreducible representations may admit a periodic action for the raising and lowering generators. Important work has already been done towards the classification of these representations [7, 8]. It seems to us that apart from the $\mathcal{U}_q(sl(2))$ [9] case there is still, however, no complete classification.

On the other hand, there are already explicit expressions for representations of $\mathcal{U}_q(sl(N))$ at roots of unity, the case we will consider from now on.

In [10–18], explicit expressions for representations with periodic (or cyclic) actions of the generators are given.

In [19–22], explicit expressions for the usual representations of $\mathcal{U}_q(sl(N))$ are written, which lead, when the deformation parameter q goes to a root of unity, to either irreducible or reducible (sometimes not totally reducible) representations, depending on their highest weight [6, 21, 23].

The irreducible sub-factors of representations that become reducible or indecomposable in the limit where q is a root of unity have sometimes no classical counterpart [6, 21] and

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we call them atypical by analogy with the case of representations of superalgebras. Some of them also appear as sub-factors of some degenerations of periodic representations [12].

In this paper, we present an improvement of [15] based on the Gelfand–Zetlin construction, which allows us to write explicitly, with the same formalism, irreducible representations of $\mathcal{U}_q(sl(N))$ at roots of unity independently of their nature (i.e. periodic, semi-periodic, usual or some atypical). Their nature is actually encoded in the generalized parameters involved in the Gelfand–Zetlin basis. All types of finite dimensional irreducible representations we are aware of enter in this scheme (however, atypical representations generally need a special treatment).

In section 2, we present the formalism and some general rules for the construction of finite dimensional irreducible representations using the adapted Gelfand–Zetlin pattern. The main types of representations are presented as examples in section 3. As an application, we finally give in section 4 a set of relations among the generators of the centre of $\mathcal{U}_q(sl(N))$ that generalizes the relations derived in [24, 25].

2. The adapted Gelfand–Zetlin basis

2.1. The quantum algebra $\mathcal{U}_q(sl(N))$

The quantum algebra $\mathcal{U}_q(sl(N))$ [1, 2] is defined by the generators k_i, k_i^{-1}, e_i, f_i ($i = 1, \ldots, N-1$) and the relations

$$\begin{aligned}
&k_i e_j = q^{a_{ij}} e_j k_i \qquad k_i f_j = q^{-a_{ij}} f_j k_i \\
&[e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}} \\
&[e_i, e_j] = 0 \qquad \text{for } |i - j| > 1 \\
&e_i^2 e_{i\pm1} - (q + q^{-1}) e_i e_{i\pm1} e_i + e_{i\pm1} e_i^2 = 0 \\
&f_i^2 f_{i\pm1} - (q + q^{-1}) f_i f_{i\pm1} f_i + f_{i\pm1} f_i^2 = 0
\end{aligned} \tag{2.1}$$

where $(a_{ij})_{i,j=1,\ldots,N-1}$ is the Cartan matrix of $sl(N)$, i.e. $a_{ii} = 2$, $a_{i,i\pm1} = -1$ and $a_{ij} = 0$ for $|i - j| > 1$.

We will not use the (standard) co-algebra structure in the following.

Let us now define the adapted Gelfand–Zetlin basis for the representations of $\mathcal{U}_q(sl(N))$.

2.2. Vectors of the Gelfand–Zetlin basis

The states are

$$|p\rangle = \left|\begin{array}{ccccccc}
p_{1N} & & p_{2N} & \cdots & p_{N-1,N} & & p_{NN} \\
& p_{1N-1} & & \cdots & & p_{N-1,N-1} & \\
& & \ddots & \cdots & \iddots & & \\
& & p_{12} & & p_{22} & & \\
& & & p_{11} & & &
\end{array}\right\rangle \tag{2.2}$$

(with respect to [15], we use $p_{il} = h_{il} - i$ instead of h_{il}.)

As usual, the indices of the first line are fixed on a given representation, whereas the others move by steps of ±1 under the action of the raising and lowering generators. The whole set of p_{il} is defined up to an overall constant (only differences enter in the formulae). One can constrain, for example, $\sum p_{iN}$, or p_{NN}, to be zero. The actions of the Cartan generators are diagonal on this basis.

In the classical case, and also in the quantum case when q is generic, the indices p_{il} are integer and satisfy the triangular identities

$$p_{i,l+1} \geqslant p_{il} > p_{i+1,l+1}. \tag{2.3}$$

The first line of indices determines in this case the highest weight of the representation.

In the case we consider in the following, q is a root of unity and the indices p_{il} are *complex.* Let m be the smallest integer such that $q^m = 1$. We will only consider the case of odd m in this paper. Most of the result can be adapted to the case where m is even by applying the prescription of [15].

Since the indices p_{il} appear in the expressions of the matrix elements only through the quantities $q^{p_{il}}$, they can consistently be defined modulo m for most of the representations, i.e. two states with indices differing by multiples of m can be identified. This will be our convention unless we specify it in the text.

We define the 'fractional part' $\mathrm{FP}(p_{il})$ of p_{il} by

$$\mathrm{FP}(p_{il}) = p_{il} \qquad [\mathrm{mod}\tfrac{1}{2}]. \tag{2.4}$$

Since the generators move the indices p_{il} by integer steps, their fractional part is then fixed on a representation, even if they do not belong to the first line.

The specifications and restrictions on the values of the indices p_{il} will be given after the action of the generators on the vectors $|p\rangle$.

2.3. *Action of the generators*

The action of the generators $k_l^{\pm 1}$, e_l, f_l is given by

$$\begin{aligned}
k_l^{\pm 1}|p\rangle &= q^{\pm\left(2\sum_{i=1}^{l} p_{il} - \sum_{i=1}^{l+1} p_{i,l+1} - \sum_{i=1}^{l-1} p_{i,l-1} - 1\right)}|p\rangle \\
f_l|p\rangle &= \sum_{j=1}^{l} c_{jl} \frac{P_1'(j,l;p)P_2'(j,l;p)}{P_3'(j,l;p)}|p_{jl}-1\rangle \\
e_l|p\rangle &= \sum_{j=1}^{l} c_{jl}^{-1} \frac{P_1''(j,l;p_{jl}+1)P_2''(j,l;p_{jl}+1)}{P_3''(j,l;p_{jl}+1)}|p_{jl}+1\rangle
\end{aligned} \tag{2.5}$$

where $|p_{jl} \pm 1\rangle$ denotes the state differing from $|p\rangle$ by only $p_{jl} \to p_{jl} \pm 1$, and

$$\begin{aligned}
P_1'(j,l;p) &= \prod_{i=1}^{l+1} [\varepsilon_{ij}(p_{i,l+1} - p_{j,l} + 1)]^{1-\eta_{ijl}} \\
P_1''(j,l;p_{jl}+1) &= \prod_{i=1}^{l+1} [\varepsilon_{ij}(p_{i,l+1} - p_{j,l})]^{\eta_{ijl}}
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
P_2'(j,l;p) &= \prod_{i=1}^{l-1} [\varepsilon_{ji}(p_{j,l} - p_{i,l-1})]^{\eta_{j,i,l-1}} \\
P_2''(j,l;p_{jl}+1) &= \prod_{i=1}^{l-1} [\varepsilon_{ji}(p_{j,l} - p_{i,l-1} + 1)]^{1-\eta_{j,i,l-1}}
\end{aligned} \tag{2.7}$$

$$
\begin{aligned}
P_3'(j,l;p) &= \prod_{\substack{i=1\\ i\neq j}}^{l} [\varepsilon_{ij}(p_{i,l}-p_{j,l})]^{1/2}[\varepsilon_{ij}(p_{i,l}-p_{j,l}+1)]^{1/2} \\
P_3''(j,l;p_{jl}+1) &= \prod_{\substack{i=1\\ i\neq j}}^{l} [\varepsilon_{ij}(p_{i,l}-p_{j,l}-1)]^{1/2}[\varepsilon_{ij}(p_{i,l}-p_{j,l})]^{1/2}
\end{aligned}
\tag{2.8}
$$

ε_{ij} being the sign defined by

$$
\varepsilon_{ij} = \begin{cases} 1 & \text{if } i \leqslant j \\ -1 & \text{if } i > j. \end{cases}
\tag{2.9}
$$

The parameters η_{ijl} are introduced to break the symmetry between the actions of e_l and f_l, and to allow one to vanish whereas the other does not. They will be taken to be $0, \frac{1}{2}$ (the standard value) or 1. They are not counted as 'continuous parameters' in the following. These actions of the generators on the Gelfand–Zetlin vectors define a module over $\mathcal{U}_q(sl(N))$ since they can be formally obtained from those of [15] by a change of normalization (see appendix for details).

From the expression of the above matrix elements, it is obvious that the indices p_{il} belonging to the same line l play a symmetric role. They can formally be permuted. This remark did not hold in the classical case when the indices were always related altogether by triangular identities (2.3).

Let us denote by α_i $(i = 1, \ldots, N-1)$ the simple roots of $sl(N)$, and by $\alpha_{ij} = \alpha_i + \cdots + \alpha_{j-1}$ $(i < j)$ the positive roots. We define the raising generators $e_{ij} \equiv e_{\alpha_{ij}}$ and $\tilde{e}_{ij} \equiv \tilde{e}_{\alpha_{ij}}$ for $i < j$ by

$$
\begin{cases}
e_{i,i+1} = \tilde{e}_{i,i+1} \equiv e_i & \text{for } i = 1, \ldots, N-1 \\
e_{i,j+1} = e_{ij}e_j - q^{-1}e_je_{ij} & \text{for } i < j \\
\tilde{e}_{i,j+1} = \tilde{e}_{ij}e_j - qe_j\tilde{e}_{ij} & \text{for } i < j.
\end{cases}
\tag{2.10}
$$

The lowering generators f_{ij} and $\tilde{f}_{ij}$ are defined by the same inductions.

The action of these generators on the Gelfand–Zetlin representation is given by

$$
\begin{aligned}
f_{l,l+n+1}|p\rangle &= \sum_{\substack{j_i=1,\ldots,l+i\\(i=0,\ldots,n)}} \left(\prod_{i=0}^{n} c_{j_il+i}\right)\left(\prod_{i=0}^{n-1} -\epsilon_{j_{i+1}j_i}\right) \\
&\quad \times q^{-(p_{j_nl+n}-p_{j_0l}+n)} \frac{\mathbb{P}_1'(p)\mathbb{P}_2'(p)}{\mathbb{P}_3'(p)} |p_{j_0l}-1, \ldots, p_{j_nl+n}-1\rangle \\
e_{l,l+n+1}|p\rangle &= \sum_{\substack{j_i=1,\ldots,l+i\\(i=0,\ldots,n)}} \left(\prod_{i=0}^{n} c_{j_il+i}^{-1}\right)\left(\prod_{i=0}^{n-1} \epsilon_{j_{i+1}j_i}\right) \\
&\quad \times q^{(p_{j_nl+n}-p_{j_0l})} \frac{\mathbb{P}_1''(p_{j_il+i}+1)\mathbb{P}_2''(p_{j_il+i}+1)}{\mathbb{P}_3''(p_{j_il+i}+1)} |p_{j_0l}+1, \ldots, p_{j_nl+n}+1\rangle.
\end{aligned}
\tag{2.11}
$$

For $\tilde{f}_{ij}$ and $\tilde{e}_{ij}$, just change the sign of the exponent of q.

The symbols $\mathbb{P}_a'$ and $\mathbb{P}_a''$ $(a = 1, \ldots, 3)$ denote the product of all the factors coming from the product of P_a' and P_a'' except those involving two of the modified indices p_{j_il+i} $(i = 0, \ldots, n)$. Hence, $\mathbb{P}_a'$ (respectively $\mathbb{P}_a''$) are the common factors of the products $f_l \cdots f_{l+n}$ (respectively $e_l \cdots e_{l+n}$) that arise in the expansion of $f_{l,l+n+1}$ (respectively $e_{l,l+n+1}$), and they do not depend on the order of the product. The q-numbers involving differences of the indices $p_{j_i l+i}$ $(i = 0, \dots, n)$ (which depend on the order of the product), gather and reduce to the single power of q.

Note that for real FPs of the indices, each q-bracket in the above P factors is real. Thus, for example, if all the exponents η_{ijl} are equal to $\frac{1}{2}$, one has Hermitian conjugacy $e_\alpha = f_\alpha^\dagger$. Moreover, $(k_i^{\pm 1})^\dagger = k_i^{\mp 1}$.

2.4. Restrictions on the values of the indices

The above expressions (2.5) involve P_3' and P_3'' in the denominators. They are well defined if either these denominators never vanish, or they are compensated by zeros in the numerators.

The first case is what happens for the most generic representations (i.e. generic indices p_{il}, not generic q), with maximal dimension and number of parameters [7]. In this case, any two indices of the same line p_{il} and p_{jl} have unequal fractional parts, i.e. $\mathrm{FP}(p_{il} - p_{jl}) \neq 0$. Even after translations by integers due to the action of the generators, the q-numbers $[p_{il} - p_{jl}]$ never vanish and neither does the whole denominator. The number of parameters and the dimension will be given in the examples.

On the other hand, some indices p_{il} and p_{jl} of the same line can have the same fractional part, on the condition that zeros in the numerators compensate the denominators when they vanish. This happens if some indices belonging to the adjacent lines $l \pm 1$ have the same fractional part as p_{il} and p_{jl}.

We study here a sufficient condition for the matrix elements to be well defined. It is the condition that forbids any pair of indices of the same line becoming equal under the action of raising and lowering generators. This condition leads in particular to the case of the usual q-deformed representations. Let us denote by $n_l(x)$ the number of indices p_{il} of line l with fractional part x. One part of the condition is that this function (which is non-zero only for a finite set of points $C/\frac{1}{2}\mathbb{Z}$, of course) obeys the following inequalities

$$n_{l+1}(x) - 2n_l(x) + n_{l-1}(x) \geqslant 0 \qquad \text{if } n_l(x) > 1. \tag{2.12}$$

Consider indeed two indices p_{il} and p_{jl} with the same fractional part. The action of f_l and e_l, which is to translate them by ± 1, may make them become equal. If they reach the point where $q^{(p_{il})} = q^{(p_{jl}-1)}$, then the denominator P_3' or P_3'' vanishes in the matrix element $\langle p_{jl} - 1 | f_l | p \rangle$ or $\langle p_{il} + 1 | e_l | p \rangle$, respectively. In order to keep these matrix elements finite, some factors of the numerator have to vanish also. Furthermore, the matrix elements $\langle p | e_l f_l | p \rangle$ and $\langle p | f_l e_l | p \rangle$ have to remain finite, and moreover $\langle p | e_l | p_{jl} - 1 \rangle \langle p_{jl} - 1 | f_l | p \rangle$ and $\langle p | f_l | p_{il} + 1 \rangle \langle p_{il} + 1 | e_l | p \rangle$ have to be zero in order to (i) keep the structure of the module (preserve $[e_l, f_l] = (k_l - k_l^{-1})/(q - q^{-1})$), (ii) forbid $q^{(p_{il})}$ and $q^{(p_{jl})}$ becoming equal, which would lead to further divergences. These constraints are satisfied if

$$\begin{gathered}
\sum_{\{i' | 1 \leqslant i' \leqslant l+1 \,\&\, q^{(p_{i',l+1})} = q^{(p_{jl}-1)}\}} (1 - \eta_{i'jl}) + \sum_{\{i' | 1 \leqslant i' \leqslant l-1 \,\&\, q^{(p_{i',l-1})} = q^{(p_{jl})}\}} \eta_{j,i',l-1} > \tfrac{1}{2} \\
\sum_{\{i' | 1 \leqslant i' \leqslant l+1 \,\&\, q^{(p_{i',l+1})} = q^{(p_{jl}-1)}\}} \eta_{i'jl} + \sum_{\{i' | 1 \leqslant i' \leqslant l-1 \,\&\, q^{(p_{i',l-1})} = q^{(p_{jl})}\}} (1 - \eta_{j,i',l-1}) > \tfrac{1}{2}
\end{gathered} \tag{2.13}$$

plus analogous constraints on $\eta_{i'il}$ and $\eta_{i,i',l-1}$. The sum of conditions (2.13) implies that $q^{p_{il}}$ and $q^{p_{jl}}$ are separated, on the discrete circle $\{q^{np_{il}}\}_{n=0,\dots,m-1}$ by at least two pairs of indices. If the indices were not defined modulo m, we would need at least one pair $p_{i',l'}$, $p_{i'',l''}$, with $l', l'' = l \pm 1$, with the same fractional part as p_{il} and p_{jl}, satisfying the usual triangular identities

$$\begin{cases} p_{jl} > p_{i',l'} \geqslant p_{il} & \text{if } l' = l+1 \\ p_{jl} \geqslant p_{i',l'} > p_{il} & \text{if } l' = l-1 \\ p_{jl} > p_{i'',l''} \geqslant p_{il} & \text{if } l'' = l+1 \\ p_{jl} \geqslant p_{i'',l''} > p_{il} & \text{if } l'' = l-1. \end{cases} \tag{2.14}$$

Since the indices are periodic, $q^{p_{il}}$ and $q^{p_{jl}}$ have two ways to reach each other on the circle, so two pairs of indices belonging to the adjacent lines are needed to prevent them from becoming equal, one pair for each interval $q^{p_{il}}$ and $q^{p_{jl}}$ define on the circle $\{q^{np_{il}}\}_{n=0,\dots,m-1}$.

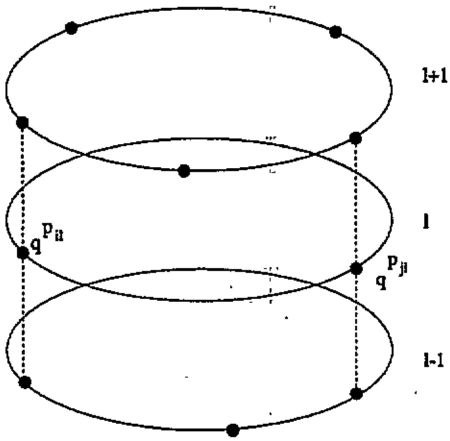

Figure 1. $q^{p_{il}}$ and $q^{p_{jl}}$ are separated, on each side they define on the circle, by two pairs of indices with the same FP from adjacent lines $l-1$ or $l+1$.

Then $n_l(x)$ indices on line l with the same fractional part x have to be separated by (at least) $2n_l(x)$ indices from lines $l \pm 1$ with the same fractional part x, which is precisely the condition expressed by (2.12). If $n_l(x) < 2$, the above discussion of course does not apply and there is no constraint at level l. If all the exponents η are equal to $\frac{1}{2}$, the condition (2.12) is enough since (2.13) are then automatically satisfied.

The set of indices with fractional part x can then be gathered into a sum of sub-triangles with a possible line (which can be broken) starting from the lowest point of the biggest triangle. For this we use the symmetry among the indices that allows each line to be reordered. The triangular shape is the most natural since it recalls the classical one with the triangular identities (2.3). With respect to [15], the new point is the possibility of a sum of triangles with the *same* fractional part, and the line continuing the biggest triangle.

Figure 2. An admissible set of indices with the same FP.

This analysis is done independently for each x, so that several sets of indices with FP x that correspond to different x can coexist.

A special case of a sub-triangle of indices with the same fractional part has to be considered: when the indices p_{jl} with $1 \leqslant j \leqslant l-N+N_1$ and $N-N_1+1 \leqslant l \leqslant N$ (since the indices of the same line play the same role, we choose here the upper left triangle for convenience) satisfy the equalities

$$p_{l,l+1} = p_{ll} = p_{l+1,l+1} + 1 \tag{2.15}$$

(similar to (2.3), but with equalities instead of inequalities), then all these indices are frozen (e_l or f_l cannot move them). All of them but $p_{1,N-N_1+1} = p_{1N}$ actually disappear from the matrix elements, since all numerators and denominators involving them systematically cancel altogether (the exponents η related to pairs of indices of the sub-triangle are chosen to be $\frac{1}{2}$). Only the terms $P_1'(j, N_1+1; p)$ and $P_1''(j, N_1+1; p)$ of the numerators still involve $p_{1,N-N_1+1} = p_{1N}$. In the actions of e_l or f_l (2.5), the terms corresponding to shifts of indices of this triangle can be removed. This simplification allows us to forget completely about the existence of these indices, except p_{1N} (fixed) of the first line, that appears in $P_1'(j, N_1+1; p)$ and $P_1''(j, N_1+1; p)$. The indices of the triangle are no longer taken into account in the function $n_l(x)$. The existence of p_{1N} in $P_1'(j, N_1+1; p)$ and $P_1''(j, N_1+1; p)$, however, changes equation (2.12) at level N_1+1 to

$$n_{N_1+2}(p_{1N}) - 2n_{N_1+1}(p_{1N}) + n_{N_1}(p_{1N}) + 1 \geqslant 0 \qquad \text{if } n_{N_1+1}(x) > 1. \tag{2.16}$$

By an abuse of notation, we write $n_l(p_{1N})$ instead of $n_l(\mathrm{FP}(p_{1N}))$. This should not lead to any confusion.

Again, several such sub-triangles can coexist.

3. Examples

3.1. Periodic (cyclic) representations

The most generic representations (do not confuse with generic q) are, as explained above, those for which $n_l(x) < 2$ for all l and x, i.e. two indices of the same line do not have the same fractional part. The indices are also not bounded from above or below by other indices of adjacent lines. The dimension is the maximal allowed dimension when q is a mth root of unity, i.e., for m odd, $(m)^{N(N-1)/2}$. Each of the $N(N-1)/2$ indices $p_{il}(l < N)$ takes m values. (For m even, a case we do not consider here, the representation would not be irreducible unless we identify $|p_{il} + m/2, p_{jl} + m/2\rangle$. In this case, the dimension is given in [15].)

These representations are called periodic (or cyclic) since for α a positive root, f_α^m and e_α^m act as (generally non-zero) scalars on them.

The continuous parameters for periodic representations are

- the c_{jl}, for $1 \leqslant j \leqslant l < N$
- the p_{jl}, for $1 \leqslant j \leqslant l \leqslant N$ (in fact only by their qth power, and modulo integer powers of q for those which do not belong to the top line).

The total number of parameters is then N^2-1 (after taking into account the constraint $\sum p_{iN} = 0$ or $p_{NN} = 0$). These parameters are indirectly related to the values of the N^2-1 central operators f_α^m, e_α^m and k_i^m. The values of the q-deformed ordinary Casimir operators are actually not independent of those (see section 4). Both the dimension and number of parameters agree with [7]. The values of η_{ijl} do not matter in this case.

All the other examples correspond to less generic cases.. For non-generic representations, the parameters live on sub-manifolds of the N^2-1 dimensional manifold of the whole set of parameters (for instance fractional parts of some indices become equal). This possibly leads to

- loss of periodicity of some f_α or e_α
- reduction of dimension.

3.2. Semi-periodic (semi-cyclic) representations

Semi-periodic representations are highest weight representations for which the lowering generators are periodic (respectively lowest weight representations with periodic raising generators).

Take $n_l(x) \leqslant 1 \quad \forall l, x$, and $\text{FP}(p_{il} - p_{i,l+1}) = 0$ for $i \leqslant l$. Choose $\eta_{ijl} = 1 \quad \forall i, j, l$.

Then we get a highest weight representation. The highest weight state $|p_0\rangle$ is given by $p_{il} = p_{iN} \quad \forall i \leqslant l \leqslant N$. On this state,

$$e_l|p_0\rangle = 0 \qquad \forall l < N \qquad \text{and hence } e_\alpha|p_0\rangle = 0 \tag{3.1}$$

for all the raising generators e_α.

The dimension of these representations is the same as for periodic representations, and the number of parameters is $(N-1)(N+2)/2$ (i.e. $N-1$ independent p_{iN} and $N(N-1)/2$ c_{jl}). The f_α remain periodic on these representations; for this reason we call them semi-periodic. The values of the central operators f_α^m and k_l^m are independent and related to the remaining parameters. Note that the vanishing of a f_α^m is not directly related to a particular equality of some FP of indices, but rather to more general algebraic equations among the $q^{mp_{il}}$ and the c_{jl}^m.

Semi-periodic representations can also be lowest weight representations, if $\eta_{ijl} = 0$. More complicated examples with mixed vanishing of e_α^m and f_β^m exist, which can be obtained directly by a suitable choice of the parameters, or also by braiding action of the Weyl group [26] on a highest weight semi-periodic representation.

This example of representations could not be taken into account in our first approach [15], because the symmetry between raising and lowering generators was not broken enough.

3.3. Nilpotent representations

Nilpotent representations are representations with a highest weight vector and a lowest weight vector, and hence nilpotent action of all the raising and lowering generators. They still have complex parameters related to the values of the operators k_l^m.

Take as before $n_l(x) \leqslant 1 \quad \forall l, x$, and $\text{FP}(p_{il} - p_{i,l+1}) = 0$ for $i \leqslant l$. Choose now $\eta_{ijl} = \frac{1}{2} \quad \forall i, j, l$.

The dimension is in this case $(m)^{N(N-1)/2}$, i.e. the same as for the periodic representation. This representation is nilpotent (i.e. $f_\alpha^m = e_\alpha^m = 0$ for every positive root α) and it is not necessary to consider the indices p_{il} modulo m, since the range of values for each index is bounded above and below by indices of adjacent lines, the upper ones being fixed. The nilpotent representation is characterized by $N-1$ parameters (the p_{iN} with $\sum p_{iN} = 0$ or $p_{NN} = 0$), corresponding to the values of the operators k_l^m or to the q-deformed usual Casimir operators. (The parameters c_{jl} can here be set to one by a change of normalization.)

We have the same highest weight vector as for the semi-periodic representations. But the f_α are no longer periodic. The specification $p_{il} = p_{i,l+1} - m + 1 \quad \forall i \leqslant l \leqslant N$ indeed defines the lowest weight vector of this representation.

In [15], we did not have the correct number of parameters for this kind of representation.

3.4. Usual q-deformed representations

We now consider usual representations, i.e. those that are q-deformations of the classical representations, in the limit where q is a root of unity. In the classical case, or when q is not a root of unity, the Gelfand–Zetlin indices are integers and there is no reason to define them modulo m. When we take the limit where q is a root of unity, we expect no periodicity of the indices (usual representations are highest-weight and lowest-weight ones) so they are not considered modulo m.

The usual representations correspond in the Gelfand–Zetlin formalism [19] to the usual choice where $n_l(0) = l$ for all $l \leqslant N$, i.e. all the indices p_{il} have the same fractional part 0. The exponents η_{ijl} are set to $\frac{1}{2}$. *No continuous parameter* survives since the c_{jl} can be absorbed in a change of normalization.

Only a finite number of representations, those with a highest weight satisfying [6, 23, 15]

$$p_{1N} - p_{NN} \leqslant m \tag{3.2}$$

are well defined in the Gelfand–Zetlin formalism for $q^m = 1$. The condition (3.2) expresses the fact that the qth powers of the indices of the first line do not wind more than exactly once around the circle $\{q^n\}_{n=0,\dots,m-1}$. This condition is also the unitarity condition, i.e. all the matrix elements of e_l and f_l are real for the usual representations, and the matrices of k_l are unitary. Furthermore, the matrix representing e_l is the transposed matrix of that of f_l.

The representations with highest weights that do not obey (3.2) are considered in the following subsection, although they are not all atypical.

3.5. Atypical representations

We consider here the quantum analogue of classical (highest weight and lowest weight) irreducible representations with a highest weight that does not obey (3.2). When $q^m = 1$ these representations are not always irreducible, since some new singular vectors arise in the corresponding Verma modules [6], that are not obtained from the highest weight vector by action of the translated Weyl group. Quotienting by the sub-representation generated by these singular vectors leads to new irreducible representations that we call atypical by analogy with the case of superalgebras.

The Gelfand–Zetlin basis in the form we consider is not yet totally adapted for atypical representation. This has to be compared with the fact that, for superalgebras, the atypical representations are more difficult to describe with the Gelfand–Zetlin than the typical ones: the atypical representations of some superalgebras or quantum superalgebras were obtained, for example, in [27, 28] in the case of $gl(n|1)$ and in [29] in the case of $\mathcal{U}_q(gl(2|2))$, but the general case has not yet been published.

It seems here that a further adaptation of the Gelfand–Zetlin basis to the atypical case is possible. We already obtained some examples of (reducible or irreducible) representations that do not obey (3.2). A general study of this case will be the subject of another work. Note that the formalism of [21], in which the matrix elements do not contain divergences, provides the atypical representations of $\mathcal{U}_q(sl(3))$.

Some atypical representation can also be obtained as degenerations of periodic representations, by taking the appropriate limit of the parameters. Consider as explained before the possibility of 'freezing' a sub-triangle of indices p_{jl} with $1 \leqslant j \leqslant l - N + N_1$ and $N - N_1 + 1 \leqslant l \leqslant N$. Remember that these indices are not taken into account in $n_l(p_{1N})$, and that the inequality (2.16) holds instead of (2.12) for $x = \text{FP}(p_{1N})$ and $l = N_1 + 1$.

Choosing $n_l(p_{1N}) = 0$ for $l = 1, \dots, N_1$ leads only to representations described in the next subsection as 'partially periodic'. Some of the generators indeed remain periodic. With

$N_1 = N - 1$, they are 'flat', i.e. the multiplicities of their weights are always 1 [15, 30].

Choosing $n_l(p_{1N}) \neq 0$ can, however, lead to some cases of atypical representations. In particular, with $n_l(p_{1N}) = 1$ and $N_1 = N - 1$, we get truncated flat representations [12] that can also be seen as an irreducible part of the limit when $q^m = 1$ of representations with $p_{1N} - p_{NN} = m + 1$, i.e. just after the limit given by (3.2) [6, 21, 23].

We can recall as examples the cases of the representations of $\mathcal{U}_q(sl(3))$ of dimensions 7 for $m = 3$ and 18 or 19 for $m = 5$. A formula for the dimensions of these representations in the case of $\mathcal{U}_q(sl(3))$ is, with $N_1 = 2$ and $p_{33} = 0$,

$$\begin{aligned} d = m^2 - d_1 - d_2 \qquad & \begin{cases} d_1 = \frac{1}{2} p_{13}(p_{13} - 1) \\ d_2 = \frac{1}{2}(m - p_{13} + 1)(m - p_{13}) \end{cases} \\ = d'_1 - d'_2 \qquad & \begin{cases} d'_1 = \frac{1}{2}(m+1) p'_{23}(m + 1 - p'_{23}) \\ d'_2 = \frac{1}{2}(m-1)(p'_{23} - 1)(m - p'_{23}). \end{cases} \end{aligned} \tag{3.3}$$

The first expression corresponds to the truncation of the flat representation of dimension m^2 by its two triangular sub-factors of dimensions d_1 and d_2 (see the figure in [12]). The second expression corresponds to the same representation seen as the irreducible part of the limit when $q^m = 1$ of the representation of dimension d'_1 with first line of indices (or highest weight) $|p'_{13} = m + 1, p'_{23} = p_{13}, p'_{33} = 0\rangle$ violating (3.2) by 1; d'_2 being the dimension of its sub-representation characterized by $|p'_{13} = m, p'_{23} = p_{13}, p'_{33} = 1\rangle$. The second expression is a particular case of those classified in [6].

The generalization of these cases to $\mathcal{U}_q(sl(N))$ with $N_1 = N - 1$ is straightforward (flat representations). Different values for N_1 provide other interesting examples.

3.6. Partially periodic representations

First note that, since the whole set of indices p_{il} is defined up to an overall constant, the case of the usual q-deformed representations can be written with $n_l(x) = l$ for any given value $x \in C$ instead of $x = 0$.

As in [15, section 3], one can put in some sub-triangles (those defined by (2.12) and figure 2) of sizes $N_1, \ldots, N_a$, with $N_1 + \cdots + N_a \leqslant N$, the indices corresponding to the usual representations of some $\mathcal{U}_q(sl(N_1)), \ldots, \mathcal{U}_q(sl(N_a))$.

This prescription reduces the dimension with respect to the maximal one. Each triangle indeed contributes to the dimension by a factor equal to the dimension of the related usual representation of the corresponding $\mathcal{U}_q(sl(N_i))$, instead of a factor equal to $(m)^{N_i(N_i-1)/2}$. The number of parameters is also reduced, since all the indices of a given sub-triangle have the same fractional part, whereas the corresponding c_{jl} are 1.

The atypical representations of smaller $\mathcal{U}_q(sl(N_i))$ can also be used as the usual ones to construct partially periodic representations $\mathcal{U}_q(sl(N))$.

4. Application: a set of relations in the centre of $\mathcal{U}_q(sl(N))$

The centre of $\mathcal{U}_q(sl(N))$ is generated by the operators f_α^m, e_α^m, k_i^m, and the q-deformed classical Casimirs C_i [7]. Let us introduce supplementary Cartan generators $k_{\pm\epsilon_i}$, for $i = 1, \ldots, N$, that are needed to write the q-deformed Casimirs. These generators are such that $k_{\epsilon_i} k_{-\epsilon_{i+1}} = k_{\alpha_i} \equiv k_i$ and $\prod_{i=1}^N k_{\epsilon_i} = 1$. They satisfy the relations

$$k_{\epsilon_i} e_j k_{\epsilon_i}^{-1} = q^{\delta_{ij} - \delta_{i-1,j}} e_j \qquad k_{\epsilon_i} f_j k_{\epsilon_i}^{-1} = q^{-\delta_{ij} + \delta_{i-1,j}} f_j. \tag{4.1}$$

By convention, $k_{\zeta+\xi} = k_\zeta k_\xi$. The operators $k_{\pm\epsilon_i}^m$ are also central.

A set of generators for the deformed classical centre (i.e. the whole centre when q is not a root of unity) is given by

$$\left\{ \mathcal{C}_i = h^{-1}\left(\sum_{1\leqslant j_1<\cdots<j_i\leqslant N} k_{2\epsilon_{j_1}}\cdots k_{2\epsilon_{j_i}} \right) \right\}_{i=1,\ldots,N-1} \tag{4.2}$$

where h is the Harish–Chandra isomorphism [31, 8] between the q-deformed classical centre and the algebra of symmetric polynomials in the $k_{2\epsilon_i}$.

This isomorphism h can be written as $h = \gamma^{-1} \circ h'$, with the following notations: h' is the projection on $\mathcal{U}^0$, within the direct sum $\mathcal{U} = \mathcal{U}^0 \oplus (\mathcal{U}^-\mathcal{U} + \mathcal{U}\mathcal{U}^+)$, with $\mathcal{U} \equiv \mathcal{U}_q(sl(N))$, and where $\mathcal{U}^0$ (respectively $\mathcal{U}^+$ and $\mathcal{U}^-$) is the sub-algebra of $\mathcal{U}_q(sl(N))$ generated by the $k_{\pm\epsilon_i}$ (respectively e_i and f_i). γ is the automorphism of $\mathcal{U}^0$ given by $\gamma(k_{2\epsilon_i}) = q^{N+1-2i}k_{2\epsilon_i}$.

Let us write $\mathcal{C}_i$ as a function (actually a non-commuting polynomial) of the parameter q and the generators

$$\mathcal{C}_i = \mathcal{F}_i(q, k_{2\epsilon_j}, \lambda e_\alpha, \lambda f_\alpha) \qquad j = 1, \ldots, N, \qquad \alpha \in \text{set of positive roots} \tag{4.3}$$

with $\lambda = q - q^{-1}$.

Then the comparison of the actions of the mth powers of the generators with the actions of the generators themselves on a periodic representation, provides us with the relations that hold in the centre of the algebra:

$$\mathcal{P}_{i,m}^{(N)}(\mathcal{C}_1, \ldots, \mathcal{C}_{N-1}) = \mathcal{F}_i(q^m = 1, k_{2\epsilon_j}^m, \lambda^m e_\alpha^m, \lambda^m f_\alpha^m) \tag{4.4}$$

where $\mathcal{P}_{i,m}^{(N)}$ is a polynomial such that

$$\mathcal{P}_{i,m}^{(N)}(h(\mathcal{C}_1), \ldots, h(\mathcal{C}_{N-1})) = \sum_{1\leqslant j_1<\cdots<j_i\leqslant N} k_{2\epsilon_{j_1}}^m \cdots k_{2\epsilon_{j_i}}^m. \tag{4.5}$$

(See [25] for details and a proof of these relations for $i = 1$ or $N - 1$. A proof of these relations for $i = 2, \ldots N - 2$ will be given elsewhere.)

In (4.4), the left-hand side is a polynomial in the q-deformed classical Casimirs, whereas the right-hand side is a function of operators that are central only when q is a root of unity. The nice feature is that this function is, up to numerical coefficients, the same as the polynomial that defines the ith Casimirs in terms of the generators.

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Appendix

We give here the transformation that relates the representation given by (2.5)–(2.8) to the standard q-deformed Gelfand–Zetlin basis [19]. Let us denote by $|p)$ the vectors of the representation of $\mathcal{U}_q(sl(N))$ in the Gelfand–Zetlin basis with all the exponents η equal to

$\frac{1}{2}$, as defined in [19]. For generic values of the indices p_{il}, we define the new basis

$$
\begin{aligned}
|p) &= \lambda(p)|p\rangle \\
\lambda(p) &= \prod_{l=1}^{N-1}\prod_{j=1}^{l}\lambda_{jl}(p) \\
\lambda_{jl}(p) &= \prod_{i=1}^{l+1}(\Gamma_q(\epsilon_{ij}(p_{i,l+1}-p_{jl}+\tfrac{1}{2})+\tfrac{1}{2}))^{\epsilon_{ij}(\eta_{ijl}-1/2)} \\
&\quad \prod_{i=1}^{l-1}(\Gamma_q(\epsilon_{ji}(p_{j,l}-p_{i,l-1}+\tfrac{1}{2})+\tfrac{1}{2}))^{\epsilon_{ji}(\eta_{j,i,l-1}-1/2)}
\end{aligned}
\tag{A1}
$$

where the function Γ_q obeys

$$\Gamma_q(x+1) = [x]\Gamma_q(x). \tag{A2}$$

This function is a straightforward adaptation of the definition of [32], in which the definition of q-number is different. This transformation is well defined when the FPs of the indices p_{il} are unequal. It works formally on infinite dimensional representations with no identification of the indices modulo m. The actions of the generators on this basis are given by (2.5), which then defines a module on $\mathcal{U}_q(sl(N))$. Quotienting then by the identification of the indices modulo m, and exploration of the parameter manifold leads to the examples of representations described in this paper.

References

[1] Drinfeld V G 1986 Quantum groups *Proc. Int. Congress of Mathematicians (Berkeley, CA)* vol 1 (New York: Academic) p 798

[2] Jimbo M 1985 q-difference analogue of $\mathcal{U}(\mathcal{G})$ and the Yang Baxter equation *Lett. Math. Phys.* **10** 63

[3] Faddeev L D, Reshetikhin N Yu and Takhtajan L A 1990 Quantization of Lie groups and Lie algebras *Leningrad Math. J.* **1** 193

[4] Rosso M 1988 Finite dimensional representations of the quantum analogue of the enveloping algebra of a complex simple Lie algebra *Commun. Math. Phys.* **117** 581

[5] Lusztig G 1990 Quantum groups at roots of 1 *Geom. Ded.* **35** 89

[6] Dobrev V K 1991 *Proc. Int. Group Theory Conference (St Andrews, 1989) (London Math. Soc. Lect. Notes Series 159)* vol 1, ed Campbell and Robertson (Cambridge: Cambridge University Press); 1991 Representations of quantum groups *Proc. Symmetries in Science V: Algebraic Structures, their Representations, Realizations and Physical Applications (Schloss Hofen, Austria, 1990)* ed B Gruber, L C Biedenharn and H-D Doebner (New York: Plenum) pp 93–135

[7] De Concini C and Kac V G 1990 Representations of quantum groups at roots of 1 *Progress Math.* **92** 471

[8] De Concini C, Kac V G and Procesi C 1992 Quantum coadjoint action *J. Am. Math. Soc.* **5** 151; 1993 Some remarkable degenerations of quantum groups *Commun. Math. Phys.* **157** 405

[9] Roche P and Arnaudon D 1989 Irreducible representations of the quantum analogue of $SU(2)$ *Lett Math. Phys.* **17** 295

[10] Sklyanin E K 1983 Some algebraic structures connected with the Yang–Baxter equation. Representations of quantum algebras *Funct. Anal. Appl.* **17** 273

[11] Babelon O 1984 Represnetations of the Yang–Baxter algebrae associated to Toda field theory *Nucl. Phys.* B **230** (FS10) 241

[12] Arnaudon D 1990 Periodic and flat irreducible representations of $SU(3)_q$ *Commun. Math. Phys.* **134** 523

[13] Date E, Jimbo M, Miki K and Miwa T 1990 Cyclic representations of $\mathcal{U}_q(sl(n+1,C))$ at $q^N=1$ *RIMS* **27** 347

[14] Date E, Jimbo M, Miki K and Miwa T 1991 Generalized chiral Potts models and minimal cyclic representations of $\mathcal{U}_q(\hat{gl}(n,C))$ *Commun. Math. Phys.* **137** 133

[15] Arnaudon D and Chakrabarti A 1991 Periodic and partially periodic representations of $SU(N)_q$ *Commun. Math. Phys.* **139** 461

[16] Bazhanov V V, Kashaev R M, Mangazeev V V and Stroganov Yu G 1991 $(\mathbb{Z}_N\times)^{n-1}$ generalization of the chiral Potts model *Commun. Math. Phys.* **138** 393

[17] Chari V and Pressley A Fundamental representations of quantum groups at roots of 1 *Preprint* Riverside and King's College, London

[18] Schnizer W A 1994 Roots of unity: representations of quantum groups *Commun. Math. Phys.* **163** 293

[19] Jimbo M 1986 Quantum R-matrix related to the generalized Toda system: an algebraic approach *Lecture Notes in Physics* vol 246 (Berlin: Springer) p 334

[20] Burdík Č, Havlíček M and Vančura A 1992 Irreducible highest weight representations of quantum groups $\mathcal{U}_q(gl(n,C))$ *Commun. Math. Phys.* **148** 417

[21] Burdík Č, King R C and Welsh T A 1993 The explicit construction of irreducible representations of the quantum algebras $\mathcal{U}_q(sl(n))$ *Proc. 3rd Wigner Symp. (Oxford) Classical and quantum systems* ed L L Boyle and A I Solomon (Singapore: World Scientific)

[22] Ueno K, Takebayashi T and Shibukawa Y 1989 *Lett. Math. Phys.* **18** 215

[23] Furlan P, Ganchev A C and Petkova V B 1990 Quantum groups and fusion rules multiplicities *Nucl. Phys.* B **343** 205

[24] Kerler T 1989 Darstellungen der Quantengruppen und Anwendungen *Diplomarbeit* ETH-Zurich

[25] Arnaudon D and Bauer M 1994 Polynomial relations in the centre of $\mathcal{U}_q(sl(N))$, hepth/9310030 *Lett. Math. Phys.* **30** 251

[26] Lusztig G 1990 Finite dimensional Hopf algebras arising from quantized universal enveloping algebras *J. Am. Math. Soc.* **3** 257

[27] Palev T D 1987 *Funct. Anal. Appl.* **21** 85 (English transl. 1988 *J. Math. Phys.* **29** 2589; 1989 *J. Math. Phys.* **30** 1433)

[28] Palev T D and Tolstoy V N 1991 Finite dimensional irreducible representation of the quantum superalgebra $\mathcal{U}_q[gl(n|1)]$ *Commun. Math. Phys.* **141** 549

[29] Nguyen Anh Ky and Stoilova N I Finite-dimensional representations of the quantum superalgebra $\mathcal{U}_q[gl(2|2)]$ II: nontypical representations at generic q *Preprint* IC/94/352

[30] Arnaudon D and Chakrabarti A 1991 Flat periodic representations of $\mathcal{U}_q(\mathcal{G})$ *Commun. Math. Phys.* **139** 605

[31] Rosso M 1990 Analogues de la forme de Killing et du théorème d'Harish-Chandra pour les groupes quantiques *Ann. Sci. Éc. Norm. Sup.* **23** 445

[32] Gasper G and Rahman M 1990 *Basic Hypergeometric Series* (Cambridge: Cambridge University Press)